**\subsection{The linear mixed-effects model}**

**\label{subsec:matrixLMM}**

Consider an experiment involving $n$ observations, $v$ treatment factors and $b$ block factors. The number of levels for the $i$th treatment factor and $j$th block factor are denoted by $t\_i$ and $m\_j$, $i = 1,2, \dots, v; j = 1,2,\dots, b$, respectively. Let $\bm{y}$ be an $n \times 1$ vector of responses, then the linear mixed-effects model for the experiment can be written in matrix form as

\begin{equation}\**label{eq:matrixLMM}**

\bm{y} = \mathbf{1}\mu + \X \bm{\alpha} + \Z\bm{\beta} + \bm{\epsilon},

\end{equation}

where $\mathbf{1}$ is an $n \times 1$ vector whose elements are all unity, $\mu$ denotes the grand mean of the data, and $\bm{\epsilon}\sim \mathcal{N}(0,\sigma^2 \I\_n)$ is a $n \times 1$ vector of unobserved random experimental errors, where $\I\_n$ denotes the $n \times n$ identity matrix. The subscript $n$ in $\I\_n$ is omitted for simplicity resulting in $\I$. The treatment parameter vector is defined as

\begin{equation}

\**label{eq:treatPar}**

\bm{\alpha} = (\alpha\_{11 \dots 11}, \alpha\_{11 \dots 12}, \dots, \alpha\_{1 1 \dots 1t\_v},\dots,\alpha\_{1 1 \dots t\_{v-1}t\_v},\dots,\alpha\_{t\_1 t\_2 \dots t\_{v-1}t\_v}),

\end{equation}

where $\alpha\_{f\_1 \dots f\_v}$ denotes the effects from treatment combination $f\_1 \dots f\_v$, where $f\_i = 1, \dots, t\_i; i = 1,\dots, v$. The treatment design matrix, $\X$, in (\ref{eq:matrixLMM}) defines the allocation of treatment combinations to experimental units. The dimensions of $\X$ comprise $n$ rows, corresponding to the number of observations, and $\prod^v\_{i = 1} t\_i$ columns, corresponding to the length of vector $\bm{\alpha}$. The vector of block parameters is defined as

\begin{equation}\**label{eq:block1Par}**

\bm{\beta} = (\bm{\beta}\_1, \bm{\beta}\_2, \ldots, \bm{\beta}\_b),

\end{equation}

where

\[

\bm{\beta}\_j = (\beta\_{j1}, \beta\_{j2}, \dots, \beta\_{j m\_j})

\]

and

$\beta\_{jk} \sim \mathcal{N}(0, \sigma\_j^2)$ ($j=1,2,\dots,b; k=1,2,\dots, m\_j$). The block design matrix, $\Z$, in (\ref{eq:matrixLMM}) can then be expressed as

\begin{equation}\**label{eq:block1Mat}**

\Z = [\Z\_1 \vert \Z\_2 \vert \ldots \vert \Z\_b],

\end{equation}

where $\Z\_j$ is the design matrix corresponding to $\bm{\beta}\_j$. Thus, the dimensions of $\Z$ consist of $n$ rows, corresponding again to the number of observations, and $\sum^{b}\_{j = 1} m\_j$ columns, corresponding to the sum of the length of each vector parameter, $\bm{\beta}\_j $, in the vector $\bm{\beta}$. The treatment and block design matrices presented here are shown to be essential components of the decomposition method described in this section.

**\subsection{Null decomposition using projection matrices}**

**\label{subsec:strataDecompProj}**

The vector of responses, $\bm{y}$, in (\ref{eq:matrixLMM}) spans an $n$-dimensional Euclidean space, commonly denoted by $\mathbb{R}^n$. A vector space, $\mathbb{V}$, is a \emph{subspace} of $\mathbb{R}^n$, i.e.\ $\mathbb{V} \subset \mathbb{R}^n$, if every vector in $\mathbb{V}$ is also in $\mathbb{R}^n$ \citep{Hadi1996}. The information decomposition of $\bm{y}$ involves its separation from the $\mathbb{R}^n$ space into its constituent vector subspace components. These vector subspaces correspond to what are commonly referred to as the \emph{strata} of the ANOVA, and can be mathematically expressed as

\begin{equation}

\**label{eq:vecSpace}**

\mathbb{R}^n = \mathbb{V}\_0 \oplus \mathbb{V}\_1 \oplus, \dots , \oplus \mathbb{V}\_{b} \oplus \mathbb{V}\_{q},

\end{equation}

where $\oplus$ denotes the addition operator of the vector spaces, and $\mathbb{V}\_l$ denotes the $l$th stratum that corresponds to the $l$th term of the expanded block structure formula \citep{Brien1983}. Thus, the number of terms, denoted by $q + 1$, is directly related to the number of strata, which depends on the experimental design. For example, an experiment arranged according to a row-column design generates four strata, namely ``Grand Mean'', ``Between Rows'', ``Between Columns'' and ``Within Rows and Columns''. For any experimental design, the first and last elements in (\ref{eq:vecSpace}), i.e.\ $\mathbb{V}\_0$ and $\mathbb{V}\_{q}$, always denote the vector subspaces for the grand mean and experimental error, respectively.

Since the decomposition is the separation of the known variation in the data, the variance structure of the data, $\bm{y}$, can be expressed in spectral form as

\begin{equation}

\**label{eq:strata}**

\operatorname{Var}(\bm{y}) = \sum\_{l=0}^{q} \xi\_l \Q\_l,

\end{equation}

where $\Q\_l$ is an $n \times n$ \emph{orthogonal projector} matrix which transforms $\bm{y}$ from vector space $\mathbb{R}^n$ onto the vector subspace $\mathbb{V}\_l$ (i.e.\ stratum $l$) and $\xi\_l$ is the $i$th stratum variance (i.e.\ $\xi\_i = \operatorname{Var}(\Q\_l \bm{y})$). Since the orthogonal projector is used to linear transform a vector onto a specific vector subspace, the orthogonal projector is a \emph{projection matrix}, which has the properties of being symmetric, i.e.\ $\Q\_l' = \Q\_l$, orthogonal, i.e.\ $\Q\_l\Q\_{l'} = 0; l \neq l'$, and idempotent, i.e.\ $\Q\_l^2 = \Q\_l$ \citep{Hadi1996}.

The total unadjusted sum of squares (SS) of $\bm{y}$, i.e.\ $\bm{y}'\bm{y}$, can be decomposed into $q+1$ components of the SS, i.e.\

\begin{equation}

\**label{eq:decomp}**

\bm{y}'\bm{y} = \sum\_{l=0}^{q}\bm{y}'\Q\_l\bm{y},

\end{equation}

where $\bm{y}'\Q\_l\bm{y}$ denotes the total SS in the $l$th stratum. Equations~(\ref{eq:vecSpace}), (\ref{eq:strata}) and (\ref{eq:decomp}) thus give a basic illustration of the decomposition of the data space without the treatment, which we refer to as the \emph{null decomposition}. The remainder of this section describes each step of the null decomposition, particularly how the orthogonal projector, denoted by $\Q\_l$, is computed in stratum $l$.

For any designed experiment, the initial step of null decomposition is to separate out the first term of the block structure formula, i.e.\ the grand mean, from the data vector, $\bm{y}$. Since $\mu$, in (\ref{eq:matrixLMM}), is a vector of length $1$, the grand mean vector spans a $1$-dimensional grand mean vector subspace, denoted by $\mathbb{V}\_0$ in (\ref{eq:vecSpace}). Meanwhile, the orthogonal projector, $\Q\_l$ is the projection matrix to project a vector onto vector subspace $l$, $\mathbb{V}\_l$. This vector subspace can be generated by a set of vectors or a matrix. The grand mean vector subspace, $\mathbb{V}\_0$, is generated by a $n \times n$ averaging matrix, denoted by $\K\_n$, with all elements equal to ${n}^{-1}$. The subscript $n$ in $\K\_n$ is again omitted for simplicity, resulting in $\K$. Thus, the projection of the vector $\bm{y}$ onto the grand mean vector subspace can be written as $\Q\_{0}\bm{y}$, where $\mP\_{\K}\bm{y}$ denotes the projection matrix of matrix $\K$ and is computed as

\[

\mP\_K = \K(\K'\K)^{-1}\K'.

\]

Since $\mP\_{\K}$ can be shown be the same as $\K$, for simplicity, $\mP\_{\K} \bm{y}$ is rewritten as $\K \bm{y}$.

The next step is to obtain the orthogonal complement of $\K\bm{y}$ by subtracting the $\K\bm{y}$ from $\bm{y}$, i.e.\

\[\bm{y} - \K\bm{y} = (\I-\K)\bm{y},\]

where $(\I-\K)\bm{y}$ denotes the \emph{mean corrected observational vector}, which spans $\mathbb{V}^{\perp}\_0$ with dimension of $(n - 1)$. The vector subspace $\mathbb{V}^{\perp}\_0$ is also the orthogonal complement of $\mathbb{V}\_0$ which is spanned by $\K\bm{y}$. Furthermore, the adjusted total SS is computed by pre-multiplying the $(\I-\K)\bm{y}$ by its transpose, i.e.\

\begin{equation}

\**label{eq:adjustSS}**

[(\I-\K)\bm{y}]'[(\I-\K)\bm{y}] = \bm{y}'(\I-\K)\bm{y}.

\end{equation}

The total adjusted SS can also be computed by subtracting the grand mean of data, $\bm{y}'\K\bm{y}$, from the total unadjusted total SS, $\bm{y}'\bm{y}$, i.e.\

\[

\bm{y}'\bm{y} - \bm{y}'\K\bm{y} = \bm{y}'(\I-\K)\bm{y}.

\]

The vector $(\I-\K)\bm{y}$ is then projected onto the vector subspace of stratum $1$, i.e.\ $\mathbb{V}\_1$, which results in $\Q\_{1}\bm{y}$, where $\Q\_{1}$ is the orthogonal projector of the stratum that corresponds to the second term of the expanded block structure formula. The vector subspace, $\mathbb{V}\_1$, is generated by the block design matrix corresponding to the second term of the block structure formula, denoted by $\Z\_1$. The orthogonal projector $\Q\_{1}$ is thus given by the $\I-\K$ pre-multiplied by $\mP\_{\Z\_1}$, i.e.\ $\mP\_{\Z\_1}(\I-\K)$, which can be re-written as

\begin{equation}\**label{eq:projectB}**

\mP\_{\Z\_1}(\I-\K)\bm{y} = (\mP\_{\Z\_1} - \K)\bm{y},

\end{equation}

where vector $(\mP\_{\Z\_1} - \K)\bm{y}$ represents estimates of effects associated with the second term of the expanded block structure formula. The orthogonal complement of $(\mP\_{\Z\_1} - \K)\bm{y}$ is then derived by subtraction from the vector $(\I-\K)\bm{y}$ as

\begin{equation}

\**label{eq:orthComp}**

(\I-\K)\bm{y}- (\mP\_{\Z\_1} - \K)\bm{y} = (\I -\mP\_{\Z\_1})\bm{y},

\end{equation}

which corresponds to the elimination of the effects from the second term of the expanded block structure formula.

The SS are derived by pre-multiplying the vectors in (\ref{eq:projectB}) and (\ref{eq:orthComp}) by their transpose, as described in (\ref{eq:adjustSS}), i.e.\

\[

\bm{y}'(\I-\K)\bm{y}- \bm{y}'(\mP\_{\Z\_1} - \K)\bm{y} = \bm{y}'(\I -\mP\_{\Z\_1})\bm{y}.

\]

If the expanded block structure formula contains additional terms, the vector $(\I - \mP\_{\Z\_1})\bm{y}$ is further projected onto the next vector subspace, $\mathbb{V}\_2$, which corresponds to the third term of the expanded block structure formula. Thus, in general, the projection of the data vector, $\bm{y}$ from $\mathbb{V}\_{l}$ onto $\mathbb{V}\_{l + 1}$ can be written as $\mP\_{\Z\_{l+1}}\Q\_{l}\bm{y}$. The orthogonal complement of $\mP\_{\Z\_{l+1}}\Q\_{l}\bm{y}$ can be derived by subtraction from $\Q\_{l}\bm{y}$, i.e.\

\begin{equation}

\**label{eq:orthCompSummary}**

\Q\_{l}\bm{y}- \mP\_{\Z\_{l+1}}\Q\_{l}\bm{y} = (\I -\mP\_{\Z\_{l+1}})\Q\_{l}\bm{y} = \Q\_{l+ 1}\bm{y}, \; (l = 0,1,\dots, b, b+1; \Q\_0 = \K).

\end{equation}

The SS thus are derived by pre-multiplying the vector in (\ref{eq:orthCompSummary}) by its transpose, as follows:

\[

\bm{y}'\Q\_{l}\bm{y}- \bm{y}'\Q\_{l}\mP\_{\Z\_{l+1}}\Q\_{l}\bm{y} = \bm{y}'\Q\_{l}(\I -\mP\_{\Z\_{l+1}})\Q\_{l}\bm{y}=\bm{y}'\Q\_{l+ 1}\bm{y}, \; (l = 0,1,\dots, b, b+1; \Q\_0 = \K).

\]